Yiddish of the day

"Zol vaksn vi a
tsibile, mitn kop in
der erd" =

„זאָל וואַקסן ווי
אַ ציבעלע מיטן
קאָפּ אין דער ערד"

"may you grow like
an onion, with your head
in the ground" =

Direct Sums

- Common goal in mathematics
  - Decompose objects into smaller pieces.

ex i) Every integer n = product of primes

ii) Bases: Every vector is uniquely expressed as linear combo of bases vectors.

Goal: Do "this" for Vector spaces / subspaces.

Q: What would "a basis of subspaces" be?

Def: Let V be an $\mathbb{F}$-vs and $W_1, \ldots, W_n$ subspaces

Then

1) The sum (or, more leadingly span) of these subspaces is the set

$$W_1 + W_2 + \cdots + W_n := \{ w_1 + w_2 + \cdots + w_n \mid w_i \in W_i \}$$

Exc: Let $V_1 \ldots V_k$ be supsaces. Show

1) $V_1 + V_2 + \cdots + V_k$ is a subspace

2) $V_1 + \cdots + V_k$ is smallest subspace containing all the subspaces V_i

(Compare to the result about span $(v_1 \ldots v_k)$)

External Direct sum

Def: Let $V_1 \ldots V_k$ be vector spaces over $\mathbb{F}$. Then the "external" direct sum, denoted $V_1 \oplus V_2 \oplus \cdots \oplus V_k$

is the set $V_1 \times V_2 \times \cdots \times V_k$

Rmk: This is a vector space!

- Addition: $(v_1, \ldots, v_k) + (w_1, \ldots, w_k) = (v_1 + w_1, \ldots, v_k + w_k)$
  $v_i, w_i \in V_i$

- Scalar mult: $\alpha(v_1, v_2, \ldots, v_k) = (\alpha v_1, \ldots, \alpha v_k)$

- 0 vector: $(0_{V_1}, 0_{V_2}, \ldots, 0_{V_k})$

$\mathbb{F}^n = \mathbb{F} \oplus \cdots \oplus \mathbb{F} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{F} \right\}$

ex) $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \subseteq \mathbb{R}^3$ "$\mathbb{R}$" $\subseteq \mathbb{R}^3$

Rmk: We can compute the dimension of $V_1 \oplus \cdots \oplus V_k$ (under the assumption that these V_i are finite dim) using the result below

$$\left(\dim(V_1 \oplus \cdots \oplus V_k) = \dim(V_1) + \dim(V_2) + \cdots + \dim(V_k) \right)$$

Notice: For each i, we have an identification

$$V_i \xrightarrow{\sim} \{(0, \dots, v_i, 0, \dots, 0) : v_i \in V_i\} \leq V_1 \oplus \cdots \oplus V_k$$

"ie, given vector spaces $V_1, \dots, V_k$, we constructed a new vector space $V_1 \oplus \cdots \oplus V_k$ such that there is an isomorphic copy of each V_i inside this vector space"

"Internal" Direct sum

Def: Let U, W be subspaces of V. Then we say

"V is the "internal direct sum" of U, W"

if

1) $U + W = V$

2) $U \cap W = \{0_V\}$

Prop: U, W subspaces of V. TFAE

(internal) 1) V is the internal direct sum of U, W

(uniqueness) 2) Every vector $v \in V$ can be written ! as
$v = u + w$ for $u \in U$, $w \in W$

(external) 3) The map $U \oplus W \xrightarrow{\pi} V$ that sends
$(u, w) \longrightarrow u + w$ is an isomorphism

Pf) Assume (1). Since $V = U + W$ we already know that
$v = u + w$ for some $u \in U$, $w \in W$. Assume that
$v = u' + w'$ for $u', w' \in U, W$. Then $u + w = u' + w'$
and so $u - u' = w' - w$. Note $u - u' \in U \cap W = \{0\}$
$w' - w \in U \cap W = \{0\}$

So $u = u'$ and $w = w'$ ✓

Assume (2). Since any vector $v \in V$ is of the form

$u+w$ this map π is surjective

Moreover, since $v = u+w$ for ! u, w this map π is injective.

Moreover, π is linear

Assume (3). Since π is iso, its surjective. So every $v \in V$ can be expressed as $v = \pi(u,w) = u+w$. I.e. $V = U+W$

If $z \neq 0 \in U \cap W$ then $\pi(z, -z) = 0 \rightarrow\!\!\!\!\times$ since π injective

$\square$

How to get **Direct sums** ?

Def: Let $W \leq V$ be subspace. Then a **complimentary subspace** for W is another subspace W' st

$$V = W \oplus W'$$

Rmk/HW: These always exist (hint, extend bases)

- Note complimentary subspaces are not at all unique

ex) $V = \mathbb{R}^2$ $\quad W = \operatorname{span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$

Then $V = W \oplus \operatorname{span}\begin{pmatrix}0\\1\end{pmatrix}$

$V = W \oplus \operatorname{span}\begin{pmatrix}2\\1\end{pmatrix}$

$V = W \oplus \operatorname{span}\begin{pmatrix}1\\0\end{pmatrix}$

$\vdots$

$v = \begin{pmatrix}4\\3\end{pmatrix}$

$[v] = \begin{pmatrix}2\\3\end{pmatrix} + W$

$= \begin{pmatrix}4\\3\end{pmatrix} + \alpha\begin{pmatrix}1\\1\end{pmatrix}$ $\alpha \in \mathbb{R}$

Some way of dealing with them all

Recall: An Equivalence Relation on a set X (denoted $\sim$) is a relation st

1) $x \sim x \quad \forall x \in X$

2) $x \sim y \Rightarrow y \sim x$

3) $x \sim y$ and $y \sim z \Rightarrow x \sim z$

$\longrightarrow$ we denote $[x] = \{ y \in X \mid x \sim y \}$

$$X/\sim \, = \{ [x] \mid x \in X \}$$

Then there is always a surjective function

$$X \xrightarrow{\pi} X/\sim$$
$$\underline{x} \longrightarrow \underline{[x]}$$

Case we're interested in

Let $W \leq V$ subspace. Define a relation $\sim_W$ on V as

$$x \sim_W y \iff \underline{x - y \in W}$$

Prop: $\sim_W$ is an <u>ER</u>

Pf) 1) Note $\forall x \in V \quad x - x = 0 \in W$ and so $x \sim_W x$

2)
$\Big\}$ HW :)
3)

What is $\underline{[v]}$ as a set

$$\underline{[v]} = \{ y \mid v \sim_W y \}$$

$$= \{ y \mid v - y \in W \}$$

$$= \{ y \mid y = v + w \text{ for } w \in W \}$$

$$= \{ v + w \mid w \in W \}$$

notation

$$= v + W$$

Def: Denote $V/W := V/\sim_W = \{[v] \mid v \in V\}$

(Say "V mod W" or "V quotient W"
We call this the "quotient space")

Thm: We can make V/W into a vector space such that the canonical map

$$\pi_W : V \longrightarrow V/W$$

is a linear transformation
Moreover $\ker(\pi_W) = \underline{W}$

Pf) Define $[v] + [w] := [v+w]$. Need to check that this addition does not depend on choice of representitive

If $[v] = [v']$, $[w] = [w']$ does $[v+w] = [v'+w']$?

Ie is $v+w \underset{W}{\sim} v'+w'$. Note $v+w-(v'+w') = v-v'+w-w'$
$\in W$

So this addition is well defined ✓

Define $a[v] = [av]$. Again, check that this does not depend on choice of v. ✓

Note the 0 vector is $[0]$ in V/W.

(Now, check that under this + and scalar mult, V/W is a vector space)

Moreover the map $\pi : V \longrightarrow V/W$
$v \longmapsto [v]$

is linear since $\pi(v_1+v_2) = [v_1+v_2] = [v_1]+[v_2]$

$$= \pi(v_1) + \pi(v_2)$$

$$\pi(\alpha v) = [\alpha v] = \alpha [v]$$

Note $\ker(\pi) = \{ v \in V \mid \overset{[v]}{\pi(v)} = [0] \}$

$$= \{ v \in V \mid v \sim 0 \}$$

$$= \{ v \in V \mid v - 0 \in W \}$$

$$= \{ v \in V \mid v \in W \} = W$$

Remark / Warning / HW

- V/W is **NOT** a **Subspace** of V.
- **However!**

HW: $W \subseteq V$ (subspace) and W' a complimentary

Then the composite $W' \xhookrightarrow{\text{inclusion}} V \xrightarrow{\pi} V/W$

$w' \longmapsto w' \longmapsto [w']$

is an isomorphism $W' \cong V/W$

In particular: $\dim(V/W) = \underline{\dim V} - \underline{\dim W}$

This construction may be seems ad-hoc, however it is extremly natural.

Intuition: $(V/W, \pi_W)$ is the "best possible" vector space that sends W to 0.

Thm: (Universal property of the quotient) $W \subseteq V$ subspace.

Let $T: V \to Z$ be linear and suppose $W \subseteq \operatorname{Ker} T$

Then $\exists!\ \tilde{T}: V/W \to Z$ such that

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & Z \\ V/W \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

ie $T = \tilde{T} \circ \pi$

Pf) We define $\tilde{T} : V/W \to Z$ by

$$\tilde{T}([v]) = T(v)$$

Suppose $[v] = [v']$. we'll show that $\tilde{T}([v]) = \tilde{T}([v'])$

Since $[v] = [v']$ we have that $v - v' \in W$

then $T(v - v') = 0$

$$\overset{\shortparallel}{T(v) - T(v')} \implies T(v) = T(v')$$

is $\tilde{T}([v]) = \tilde{T}([v'])$

And $\tilde{T}$ is linear, since $\tilde{T}([v_1 + v_2]) = T(V_1 + V_2)$

$= T(V_1) + T(V_2)$

$= \tilde{T}([V_1]) + \tilde{T}([V_2])$

$\tilde{T}([\alpha v]) = T(\alpha v) = \alpha T(v)$

$= \alpha \tilde{T}[v]$ $\square$

Consequences

Thm: (1st isomorphism thm)
Let $T: V \to Z$ be linear. Then there is isomorphism

$$\tilde{T}: V/\ker T \simeq \operatorname{im}(T)$$

Pf) Given $T: V \to Z$ then by the thm above, we get a well defined map

well-defined by previous thm

$$\tilde{T}: V/\ker(T) \to Z$$ such that

$\tilde{T}([v]) = T(v)$. Now lets check $\tilde{T}$ is injective.

Assume $\tilde{T}([v]) = 0$

However $\tilde{T}([v]) = T(v) = 0$ ie $v \in \ker T$ so

$[v] = [0]$ in $V/\ker T$ $\square$

Cor: (Rank-Nullity thrm)

Let V be finite dim and $T: V \to Z$ linear.

Then

$$\dim V = \dim \ker T + \dim(\operatorname{im} T)$$

Pf) Let W' be a complimentary subspace of $\ker T$

Then $W' \cong V/\ker T$ by HW

$$\dim V = \dim(\ker T) + \dim(W')$$

$$\xrightarrow{\text{by HW}} \quad = \dim(\ker T) + \dim(V/\ker T)$$

$$\xrightarrow{\text{by 1st iso thm}} \quad = \dim(\ker T) + \dim(\operatorname{im} T)$$